Solutions
Math 220
HW # 5
October 29, 2018

Exercise 1.

(*a*) *Is* 52 *divisible by* 13*?*
(*b*) *Does* $5|0$*?*
(*c*) *Is* $(3k+1)(3k+2)(3k+3)$ *divisible by* 3*?*
(*d*) *Is* 29 *a multiple of* 3*?*
(*e*) *Is* -3 *a factor of* 66*?*
(*f*) *If* $n = 4k+1$*, does* 8 *divide* n^2-1*?*

Solution.

(*a*) Yes, $52 = 13 \cdot 4$.
(*b*) Yes, zero is divisible by any nonzero number.
(*c*) Yes, because $(3k+3)$ is divisible by 3.
(*d*) No, 29 is not a multiple of 3.
(*e*) Yes, -3 is a factor of 66 since it divides it.
(*f*) $n^2 - 1 = 16k^2 + 8k + 1 - 1 = 16k^2 + 8k$, so yes, $8|(n^2-1)$.

□

Exercise 2. *Let* a *and* b *be integers, with* $a \neq 0$*. Prove that if* $a|b$*, then* $a^2|b^2$*.*

Proof. Suppose $a|b$, then $b = ac$ for some $c \in \mathbb{Z}$. Then $b^2 = (ac)^2 = a^2c^2$ which shows $a^2|b^2$. □

Exercise 3. *Let* $a \in \mathbb{Z}$*. Prove that if* $3|2a$*, then* $3|a$*.*

Proof. Suppose that $3 \nmid a$. Then $a = 3k+1$ or $a = 3k+2$ for some $k \in \mathbb{Z}$.

Case 1 Assume $a = 3k+1$.

$$2a = 2(3k+1) = 6k+2 = 3(2k)+2$$

which shows that $2a$ is not divisible by 3.

Case 2 Assume $a = 3k+2$.

$$2a = 2(3k+2) = 6k+4 = 3(2k+1)+1$$

which again shows that $2a$ is not divisible by 3.

Therefore $3 \nmid 2a$. So by contraposition, we have that if $3|2a$, then $3|a$. □

Exercise 4. *Let $x \in \mathbb{Z}$. Prove that if $2|(x^2-5)$, then $4|(x^2-5)$.*

Proof. Assume that $4 \nmid (x^2-5)$. Then there are 3 cases:

Case 1: Assume $x^2-5=4k+1$. Since $2|4k$ and $2 \nmid 1$, we have $2 \nmid (4k+1)$, hence $2 \nmid (x^2-5)$.

Case 2: Assume $x^2-5=4k+3$. Since $2|4k$ and $2 \nmid 3$, we have $2 \nmid (4k+3)$, hence $2 \nmid (x^2-5)$.

Case 3: Assume $x^2-5=4k+2$, then $x^2=4k+7=4(k+1)+3$. This means that x^2, and hence x, is odd. Let $x=2y+1$, then $x^2=4k+3$ becomes

$$4y^2+4y+1=4k+3 \Leftrightarrow 4y^2+4y=4k+2 \Leftrightarrow 2y^2+2y=2k+2.$$

So the left side is even while the right side is odd, a contradiction. Hence we cannot have $x^2-5=4k+1$.

By contrapositive, we have that if $2|(x^2-5)$, then $4|(x^2-5)$. $\square$

Exercise 5. *Let $a,b,c \in \mathbb{Z}$ and assume $a^2+b^2=c^2$. Then $3|ab$.*
Hint: Use proof by contradiction. You will need some results from class to do this as well.

Proof. Assume that $a^2+b^2=c^2$ and $3 \nmid ab$. Then $3 \nmid a$ and $3 \nmid b$. Then $a^2=3k+1$ and $b^2=3l+1$ for some $k,l \in \mathbb{Z}$. Then

$$c^2=(3k+1)+(3l+1)=3(k+l)+2.$$

This means that c^2 has a remainder of 2 when divided by 3, which is a contradition. Therefore $3|ab$. $\square$

Exercise 6. *Let $a,b,n \in \mathbb{Z}$ with $n \geq 2$. Prove that if $a \equiv b \mod n$, then $a^2 \equiv b^2 \mod n$.*

Proof. Assume that $a \equiv b \mod n$. Then $n|(a-b)$. Observe that $a^2-b^2=(a+b)(a-b)$, hence we have that $n|(a^2-b^2)$, so $a^2 \equiv b^2 \mod n$. $\square$

Exercise 7. *Let $a,b,c,n \in \mathbb{Z}$ with $n \geq 2$. Prove that if $a \equiv b \mod n$ and $a \equiv c \mod n$, then $b \equiv c \mod n$.*

Proof. Assume $a \equiv b \mod n$ and $a \equiv c \mod n$, then $a-b=nk$ and $a-c=nl$ for some $k,l \in \mathbb{Z}$. Subtract the first equation from the second to get

$$\begin{aligned}(a-c)-(a-b) &= nl-nk \\ b-c &= n(l-k)\end{aligned}$$

Which shows that $b \equiv c \mod n$. $\square$

Exercise 8. *Let $m,n \in \mathbb{N}$ such that $m|n$. Prove that if a and b are integers such that $a \equiv b \mod n$, then $a \equiv b \mod m$.*

Proof. If $a \equiv b \mod n$, then $n|(a-b)$. Since $m|n$, by the transitive property of divides, we have that $m|(a-b)$. Therefore $a \equiv b \mod m$. $\square$

Exercise 9. *Let $a \in \mathbb{Z}$. Prove that $a^3 \equiv a \mod 3$.*

Proof. We do this in three cases.

Case 1: Assume $a = 3k$, then $a^3 - a = 27k^3 - 3k = 3(9k^3 - k)$, so $a^3 \equiv a \mod n$.

Case 2: Assume $a = 3k+1$, then $a^3 - a = (3k+1)^3 - (3k+1) = 27k^3 + 27k^2 + 9k + 1 - 3k - 1 = 27k^3 + 27k^2 + 6k = 3(9k^3 + 9k^2 + 2k)$, so $a^3 \equiv a \mod n$.

Case 3: Assume $a = 3k+2$, then $a^3 - a = (3k+2)^3 - (3k+2) = 27k^3 + 54k^2 + 36k + 8 - 3k - 2 = 27k^3 + 54k^2 + 33k + 6 = 3(9k^3 + 9k^2 + 2k)$, so $a^3 \equiv a \mod n$.

□

Exercise 10. *The product of any three consecutive integers is divisible by* 6*.*

Proof. This product can be written in the form $n(n+1)(n+2)$ for some $n \in \mathbb{Z}$. If n is even, then $2|n$, hence 2 divides the product. If n is odd, then $n+1$ is even, so $2|(n+1)$, hence 2 divides the product. So, no matter what, $2|n(n+1)(n+2)$. If $3|n$, then 3 divides the product. If $3 \nmid n$, then there are two possibilities. If $n = 3k+1$, then $n + 2 = (3k+1) + 2 = 3k + 3$ is divisible by 3. If $n = 3k + 2$, then $n + 1 = 3k + 3$, which is divisible by 3. Hence the product is always divisible by 3. Since the product is divisible by 2 and 3, it is divisible by $2 \cdot 3 = 6$. □